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Using Integration by Parts for Fractional Calculus to Solve Some Fractional Integral Problems

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we solve some fractional integrals by using integration by parts for fractional calculus. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of traditional calculus results.**

Keywords: **Jumarie type of R-L fractional calculus, fractional integrals, integration by parts for fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

The calculus founded by Newton and Leibniz is a very important scientific achievement in the history of mathematics. Fractional calculus was first proposed by the famous mathematician Hospital in 1695. A question is about what is $\frac{d^{1/2}x}{dx^{1/2}}$? After 124 years, Lacroix gave the right answer to this question for the first time that $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}}x^{1/2}$. However, for a long time, due to the lack of practical application, fractional calculus has not been widely used. With the development of science and technology, especially since the 20th century, the theory and application of fractional calculus began to be widely concerned. Fractional calculus has become a powerful tool to study fractional differential equations and fractional functions, and has been widely used in the research of physics, electrical engineering, viscoelasticity, control theory, biology, economics, and so on [1-12].

However, the definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [13-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus, we solve the following two α-fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha} arctan_{\alpha}(x^{\alpha})\right], \quad (1)$$

and

$$\left({}_0I_x^{\alpha}\right)\left[Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]\right], \quad (2)$$

where $0<\alpha\leq 1$. Integration by parts for fractional calculus, and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \tag{3}$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{4}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([18])**:** *If α,β,x_0,C are real numbers and $\beta\geq\alpha>0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{6}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([19])**:** If x,x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{7}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{8}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{10}$$

Definition 2.5 ([21])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{11}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \quad (12)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \quad (13)$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \quad (14)$$

Definition 2.6 ([22])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha -1}$.

Definition 2.7 ([23])**:** If $0<\alpha\le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \quad (15)$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Theorem 2.8 (integration by parts for fractional calculus) ([24])**:** *Suppose that* $0<\alpha\le 1$, a,b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]=\left[f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b}-\left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha)\otimes_\alpha\left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \quad (16)$$

III. MAIN RESULTS

In this section, we solve two fractional integrals by using integration by parts for fractional calculus.

Example 3.1: Let $0<\alpha\le 1$. Find

$$\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha arctan_\alpha(x^\alpha)\right].$$

Solution Using integration by parts for fractional calculus yields

$$\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha arctan_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^\alpha\right)\left[arctan_\alpha(x^\alpha)\otimes_\alpha\left({}_0D_x^\alpha\right)\left[\frac{1}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]\right]$$

$$=\left[\frac{1}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha arctan_\alpha(x^\alpha)\right]_0^x-\left({}_0I_x^\alpha\right)\left[\frac{1}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha -1}\right]$$

$$=\frac{1}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha arctan_\alpha(x^\alpha)-\frac{1}{2}\left({}_0I_x^\alpha\right)\left[1-\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha -1}\right]$$

$$=\frac{1}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha arctan_\alpha(x^\alpha)+\frac{1}{2}arctan_\alpha(x^\alpha)-\frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (17)$$

Example 3.2: If $0<\alpha\le 1$. Find

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]\right].$$

Solution By integration by parts for fractional calculus,

$$\left({}_0I_x^{\alpha}\right)\left[Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\right]$$

$$=\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]\right]_0^x-\left({}_0I_x^{\alpha}\right)\left[2\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\otimes_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]^{\otimes_{\alpha}-1}\right]$$

$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]-2\cdot\left({}_0I_x^{\alpha}\right)\left[1-\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]^{\otimes_{\alpha}-1}\right]$$

$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}Ln_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]+2\cdot arctan_{\alpha}(x^{\alpha})-2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\,. \tag{18}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we evaluate two fractional integrals by using integration by parts for fractional calculus. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our results are generalizations of the results in classical calculus. In the future, we will continue to study the problems in engineering mathematics and fractional differential equations.

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